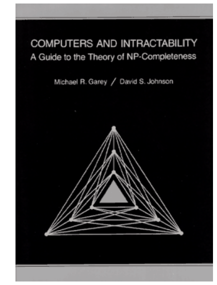
Combinatorial Search

- permutations
- backtracking
- counting
- subsets
- paths in a graph

Implications of NP-completeness



"I can't find an efficient algorithm, but neither can all these famous people."

Overview

Exhaustive search. Iterate through all elements of a search space.

Applicability. Huge range of problems (include intractable ones).

Caveat. Search space is typically exponential in size $\Rightarrow$ effectiveness may be limited to relatively small instances.

Backtracking. Systematic method for examining feasible solutions to a problem, by systematically pruning infeasible ones.

Warmup: enumerate N-bit strings

Goal. Process all 2^N bit strings of length N.

- Maintain array `a[]` where `a[i]` represents bit i.
- Simple recursive method does the job.

```
// enumerate bits in a[k] to a[N-1]
private void enumerate(int k)
{
   if (k == N)
   {  process(); return;   }
   enumerate(k+1);
   a[k] = 1;
   enumerate(k+1);
   a[k] = 0;
}
```

clean up

N = 3

0	0	0
0	0	1
0	0	0
0	1	0
0	1	1
0	1	0
0	0	0
1	0	0
1	0	1
1	0	0
1	1	0
1	1	1
1	1	0
1	0	0
0	0	0

N = 4

0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
1	1	0	1
1	1	1	0
1	1	1	1

`a[0]` `a[N-1]`

Remark. Equivalent to counting in binary from 0 to $2^N - 1$.

Warmup: enumerate N-bit strings

```
public class BinaryCounter
{
   private int N;    // number of bits
   private int[] a;  // a[i] = ith bit

   public BinaryCounter(int N)
   {
      this.N = N;
      this.a = new int[N];
      enumerate(0);
   }

   private void process()
   {
      for (int i = 0; i < N; i++)
         StdOut.print(a[i]) + " ";
      StdOut.println();
   }

   private void enumerate(int k)
   {
     if (k == N)
     {  process(); return;  }
     enumerate(k+1);
     a[k] = 1;
     enumerate(k+1);
     a[k] = 0;
   }
}
```

all programs in this lecture are variations on this theme

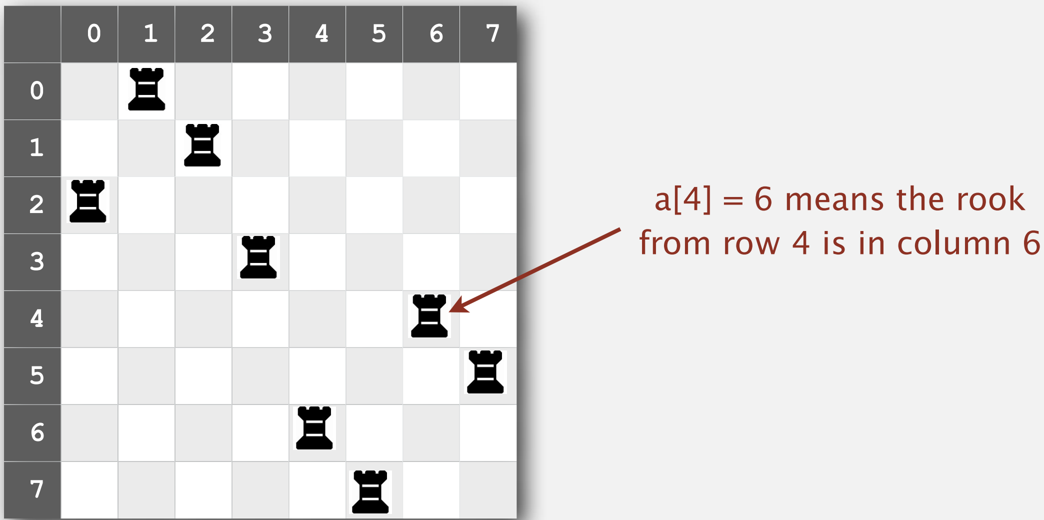
```
public static void main(String[] args)
{
   int N = Integer.parseInt(args[0]);
   new BinaryCounter(N);
}
```

```
% java BinaryCounter 4
0 0 0 0
0 0 0 1
0 0 1 0
0 0 1 1
0 1 0 0
0 1 0 1
0 1 1 0
0 1 1 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 1 1
1 1 0 0
1 1 0 1
1 1 1 0
1 1 1 1
```

- **permutations**
- backtracking
- counting
- subsets
- paths in a graph

N-rooks problem

Q. How many ways are there to place N rooks on an N-by-N board so that no rook can attack any other?




```
int[] a = { 2, 0, 1, 3, 6, 7, 4, 5 };
```

Representation. No two rooks in the same row or column $\Rightarrow$ permutation.

Challenge. Enumerate all $N!$ permutations of N integers 0 to $N - 1$.

Enumerating permutations

Recursive algorithm to enumerate all $N\,!$ permutations of N elements.

- Start with permutation `a[0]` to `a[N-1]`.
- For each value of `i`:
  - swap `a[i]` into position `0`
  - enumerate all $(N-1)\,!$ permutations of `a[1]` to `a[N-1]`
  - clean up (swap `a[i]` back to original position)

`N = 2`

```
0 1
1 0
0 1
```

`N = 3`

```
0 1 2
0 2 1
0 1 2
1 0 2
1 2 0
1 0 2
0 1 2
2 1 0
2 0 1
2 1 0
0 1 2
```

0 followed by perms of 1 2 3	1 followed by perms of 0 2 3	2 followed by perms of 1 0 3	3 followed by perms of 1 2 0
0 1 2 3	1 0 2 3	2 1 0 3	3 1 2 0
0 1 3 2	1 0 3 2	2 1 3 0	3 1 0 2
0 2 1 3	1 2 0 3	2 0 1 3	3 2 1 0
0 2 3 1	1 2 3 0	2 0 3 1	3 2 0 1
0 3 2 1	1 3 2 0	2 3 0 1	3 0 2 1
0 3 1 2	1 3 0 2	2 3 1 0	3 0 1 2

cleanup swaps that bring permutation back to original

`a[0]` `a[N-1]`

Recursive algorithm to enumerate all $N!$ permutations of N elements.

- Start with permutation `a[0]` to `a[N-1]`.
- For each value of `i`:
  - swap `a[i]` into position `0`
  - enumerate all $(N-1)!$ permutations of `a[1]` to `a[N-1]`
  - clean up (swap `a[i]` back to original position)

```
// place N-k rooks in a[k] to a[N-1]
private void enumerate(int k)
{
   if (k == N)
   {  process(); return;  }

   for (int i = k; i < N; i++)
   {
      exch(k, i);
      enumerate(k+1);
      exch(i, k);
   }
}
```

clean up

```
% java Rooks 4
0 1 2 3
0 1 3 2
0 2 1 3
0 2 3 1
0 3 2 1
0 3 1 2
1 0 2 3
1 0 3 2
1 2 0 3
1 2 3 0
1 3 2 0
1 3 0 2
2 1 0 3
2 1 3 0
2 0 1 3
2 0 3 1
2 3 0 1
2 3 1 0
3 1 2 0
3 1 0 2
3 2 1 0
3 2 0 1
3 0 2 1
3 0 1 2
```

`0` followed by perms of `1 2 3`

`1` followed by perms of `0 2 3`

`2` followed by perms of `1 0 3`

`3` followed by perms of `1 2 0`

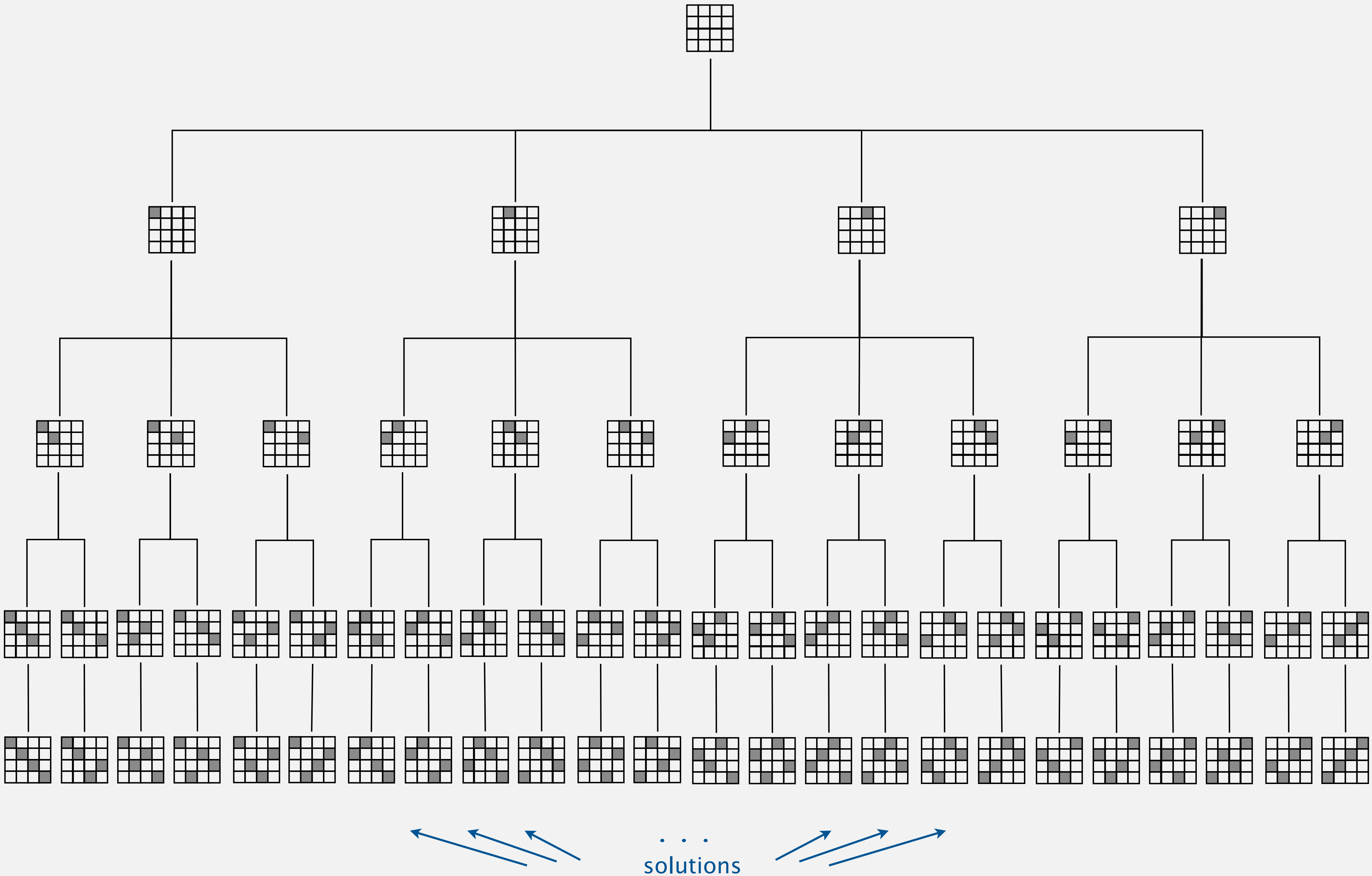
`a[0]` `a[N-1]`

Enumerating permutations

```
public class Rooks
{
   private int N;
   private int[] a;   // bits (0 or 1)

   public Rooks(int N)
   {
      this.N = N;
      a = new int[N];
      for (int i = 0; i < N; i++)
         a[i] = i;                ← initial permutation
      enumerate(0);
   }

   private void enumerate(int k)
   { /* see previous slide */  }

   private void exch(int i, int j)
   {  int t = a[i]; a[i] = a[j]; a[j] = t;   }

   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      new Rooks(N);
   }
}
```

```
% java Rooks 2
0 1
1 0

% java Rooks 3
0 1 2
0 2 1
1 0 2
1 2 0
2 1 0
2 0 1
```

4-rooks search tree



N-rooks problem: back-of-envelope running time estimate

Slow way to compute $N!$.

```
% java Rooks 7 | wc -l
5040

% java Rooks 8 | wc -l
40320

% java Rooks 9 | wc -l
362880

% java Rooks 10 | wc -l
3628800

% java Rooks 25 | wc -l
...
```

← instant

← 1.6 seconds

← 15 seconds

← 170 seconds

← forever

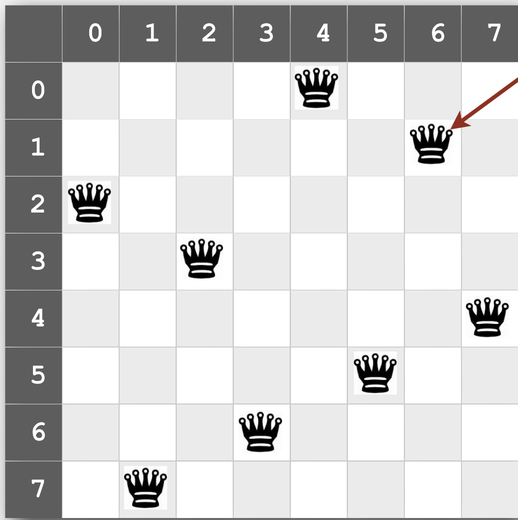
Hypothesis. Running time is about $2\,(N!\ /\ 8!)$ seconds.

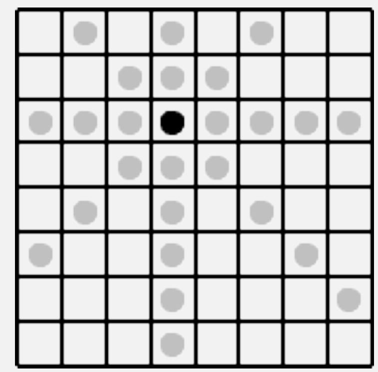
- permutations
- **backtracking**
- counting
- subsets
- paths in a graph

N-queens problem

Q. How many ways are there to place N queens on an N-by-N board so that no queen can attack any other?

a[1] = 6 means the queen from row 1 is in column 6





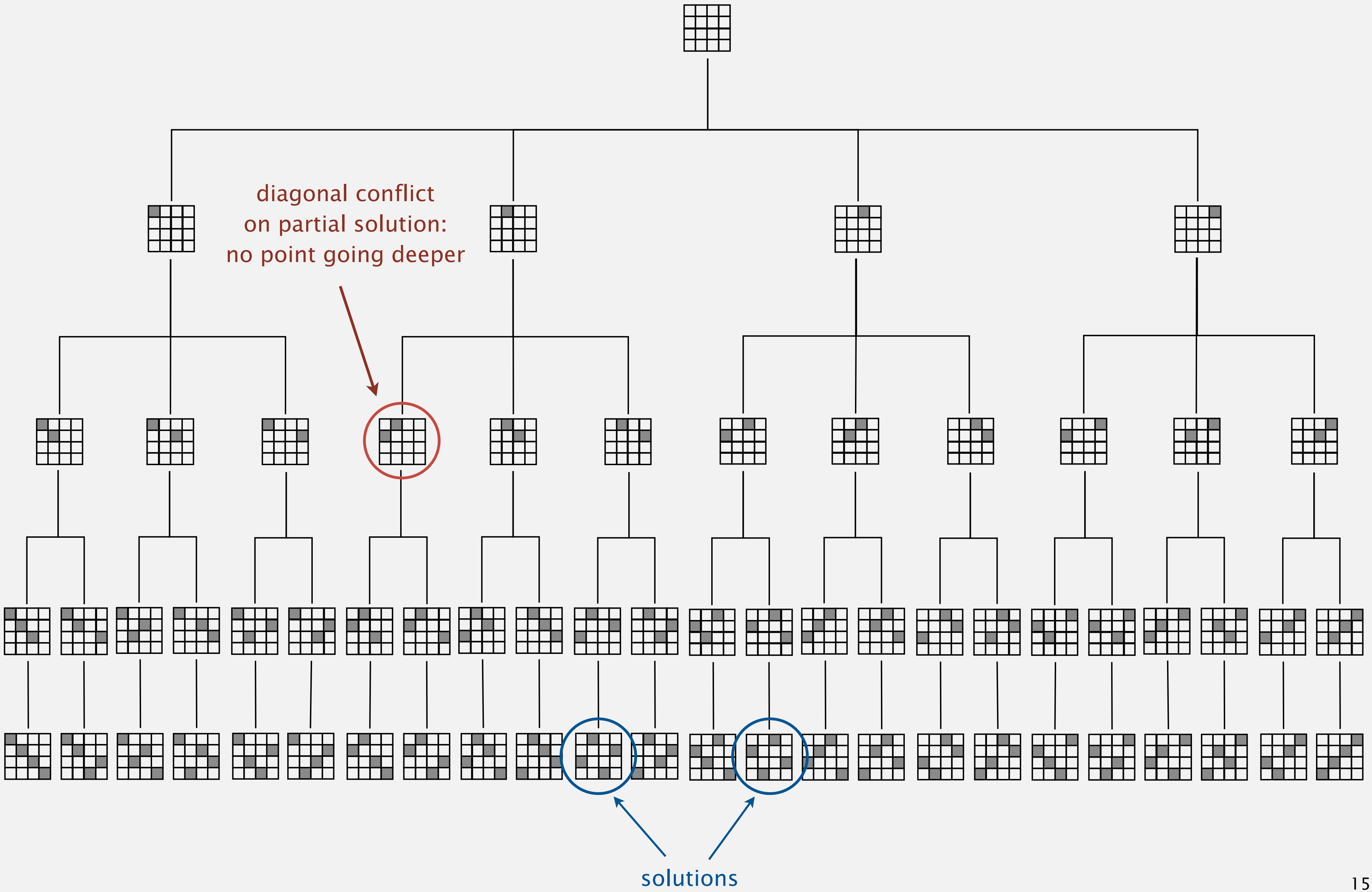
```
int[] a = { 2, 7, 3, 6, 0, 5, 1, 4 };
```

Representation. No two queens in the same row or column $\Rightarrow$ permutation.

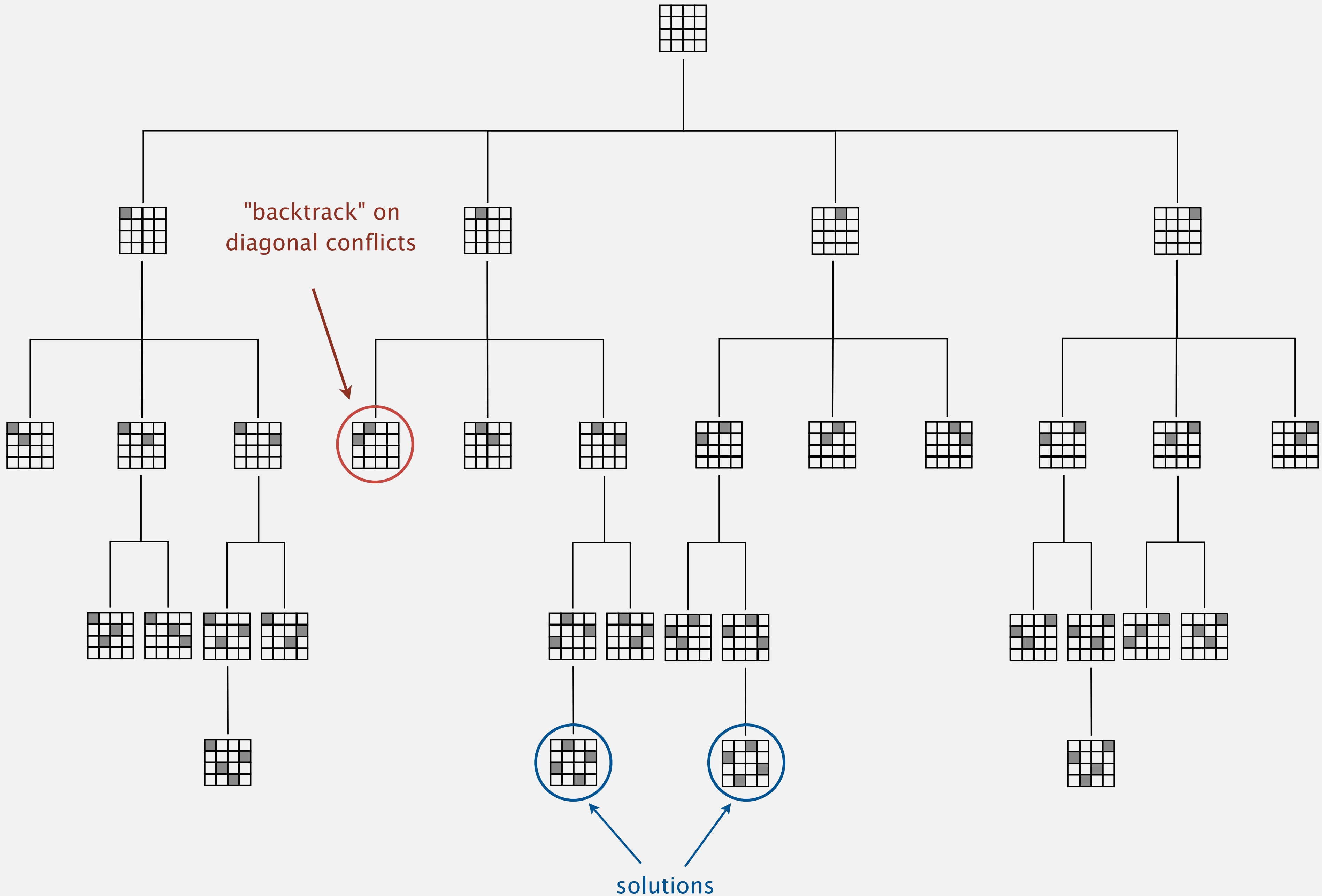
Additional constraint. No diagonal attack is possible.

Challenge. Enumerate (or even count) the solutions. ←— unlike N-rooks problem, nobody knows answer for N > 30

4-queens search tree



4-queens search tree (pruned)



N-queens problem: backtracking solution

Backtracking paradigm. Iterate through elements of search space.

- When there are several possible choices, make one choice and recur.
- If the choice is a dead end, backtrack to previous choice, and make next available choice.

Benefit. Identifying dead ends allows us to prune the search tree.

Ex. [backtracking for N-queens problem]

- Dead end: a diagonal conflict.
- Pruning: backtrack and try next column when diagonal conflict found.

N-queens problem: backtracking solution

```
private boolean canBacktrack(int k)
{
   for (int i = 0; i < k; i++)
   {
      if ((a[i] - a[k]) == (k - i)) return true;
      if ((a[k] - a[i]) == (k - i)) return true;
   }
   return false;
}

// place N-k queens in a[k] to a[N-1]
private void enumerate(int k)
{
   if (k == N)
   {  process(); return;  }

   for (int i = k; i < N; i++)
   {
      exch(k, i);
      if (!canBacktrack(k)) enumerate(k+1);
      exch(i, k);
   }
}
```

stop enumerating if adding queen k leads to a diagonal violation

```
% java Queens 4
1 3 0 2
2 0 3 1

% java Queens 5
0 2 4 1 3
0 3 1 4 2
1 3 0 2 4
1 4 2 0 3
2 0 3 1 4
2 4 1 3 0
3 1 4 2 0
3 0 2 4 1
4 1 3 0 2
4 2 0 3 1

% java Queens 6
1 3 5 0 2 4
2 5 1 4 0 3
3 0 4 1 5 2
4 2 0 5 3 1
```

a[0] a[N-1]

N-queens problem: effectiveness of backtracking

Pruning the search tree leads to enormous time savings.

N	Q(N)	N !
2	0	2
3	0	6
4	2	24
5	10	120
6	4	720
7	40	5,040
8	92	40,320
9	352	362,880
10	724	3,628,800
11	2,680	39,916,800
12	14,200	479,001,600
13	73,712	6,227,020,800
14	365,596	87,178,291,200

N-queens problem: How many solutions?

```
% java Queens 13 | wc -l
73712

% java Queens 14 | wc -l
365596

% java Queens 15 | wc -l
2279184

% java Queens 16 | wc -l
14772512

% java Queens 17 | wc -l
...
```

← 1.1 seconds

← 5.4 seconds

← 29 seconds

← 210 seconds

← 1352 seconds

Hypothesis. Running time is about $(N\,! \;/\; 2.5^N) \,/\; 43{,}000$ seconds.

Conjecture. $Q(N) \sim N\,! \;/\; c^{\,N}$, where c is about 2.54.

- permutations
- backtracking
- **counting**
- subsets
- paths in a graph

Counting: Java implementation

Goal. Enumerate all N-digit base-R numbers.

Solution. Generalize binary counter in lecture warmup.

```
// enumerate base-R numbers in a[k] to a[N-1]
private static void enumerate(int k)
{
   if (k == N)
   {  process(); return;   }

   for (int r = 0; r < R; r++)
   {
      a[k] = r;
      enumerate(k+1);
   }
   a[k] = 0;
}
```

cleanup not needed; why?

```
% java Counter 2 4
0 0
0 1
0 2
0 3
1 0
1 1
1 2
1 3
2 0
2 1
2 2
2 3
3 0
3 1
3 2
3 3

% java Counter 3 2
0 0 0
0 0 1
0 1 0
0 1 1
1 0 0
1 0 1
1 1 0
1 1 1
```

a[0] a[N-1]

Counting application: Sudoku

Goal. Fill 9-by-9 grid so that every row, column, and box contains each of the digits 1 through 9.

7		8				3		
			2		1			
5								
	4						2	6
3				8				
			1				9	
	9		6					4
				7		5		

Counting application: Sudoku

Goal. Fill 9-by-9 grid so that every row, column, and box contains each of the digits 1 through 9.

7	2	8	9	4	6	3	1	5
9	3	4	2	5	1	6	7	8
5	1	6	7	3	8	2	4	9
1	4	7	5	9	3	8	2	6
3	6	9	4	8	2	1	5	7
8	5	2	1	6	7	4	9	3
2	9	3	6	1	5	7	8	4
4	8	1	3	7	9	5	6	2
6	7	5	8	2	4	9	3	1

Solution. Enumerate all 81-digit base-9 numbers (with backtracking).

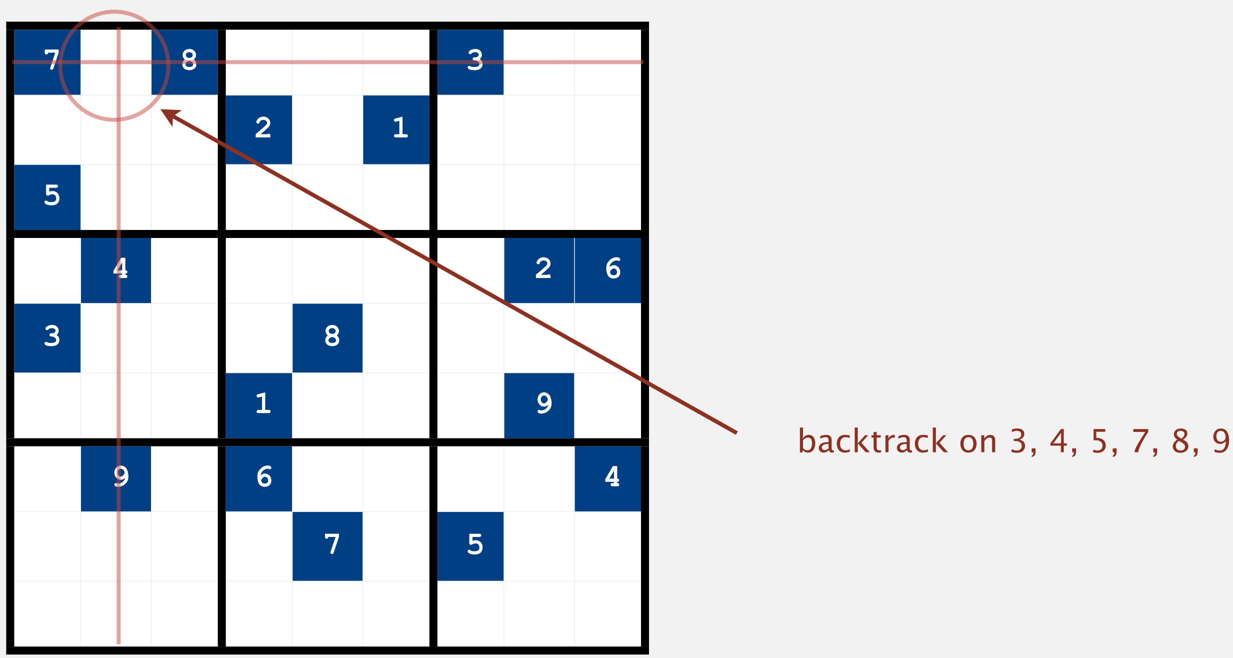
`a[]`

using digits 1 to 9 →

7		8					3		...	
0	1	2	3	4	5	6	7	8		80

Iterate through elements of search space.

- For each empty cell, there are 9 possible choices.
- Make one choice and recur.
- If you find a conflict in row, column, or box, then backtrack.



Sudoku: Java implementation

```
private void enumerate(int k)
{

   if (k == 81)
   {  process(); return;  }

   if (a[k] != 0)
   {  enumerate(k+1); return;  }

   for (int r = 1; r <= 9; r++)
   {
         a[k] = r;
         if (!canBacktrack(k))
            enumerate(k+1);
   }

   a[k] = 0;
}
```

found a solution

cell k initially filled in; recur on next cell

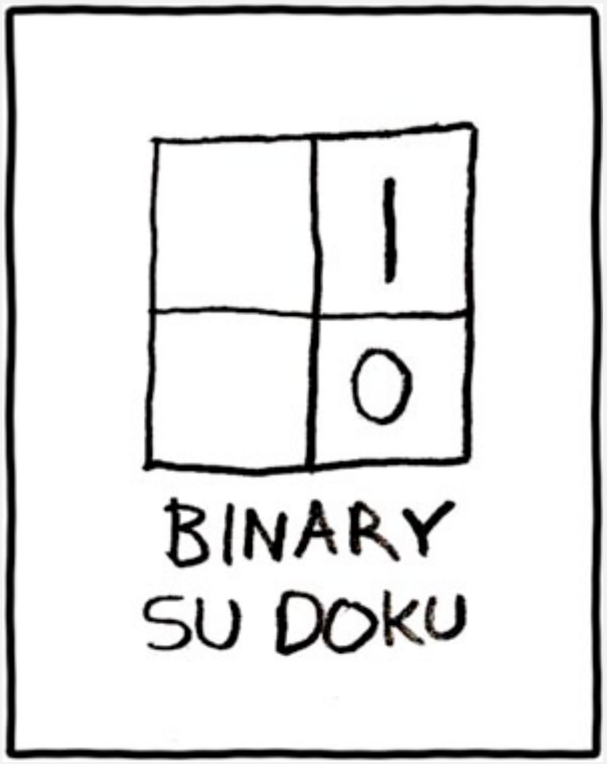
try 9 possible digits for cell k

unless it violates a Sudoku constraint (see booksite for code)

clean up

```
% more board.txt
7 0 8 0 0 0 3 0 0
0 0 0 2 0 1 0 0 0
5 0 0 0 0 0 0 0 0
0 4 0 0 0 0 0 2 6
3 0 0 0 8 0 0 0 0
0 0 0 1 0 0 0 9 0
0 9 0 6 0 0 0 0 4
0 0 0 0 7 0 5 0 0
0 0 0 0 0 0 0 0 0

% java Sudoku < board.txt
7 2 8 9 4 6 3 1 5
9 3 4 2 5 1 6 7 8
5 1 6 7 3 8 2 4 9
1 4 7 5 9 3 8 2 6
3 6 9 4 8 2 1 5 7
8 5 2 1 6 7 4 9 3
2 9 3 6 1 5 7 8 4
4 8 1 3 7 9 5 6 2
6 7 5 8 2 4 9 3 1
```

Remark. Natural generalization of Sudoku is NP-complete.



http://xkcd.com/74

- permutations
- backtracking
- counting
- **subsets**
- paths in a graph

Enumerating subsets: natural binary encoding

Given N elements, enumerate all 2^N subsets.

- Count in binary from 0 to $2^N - 1$.
- Maintain array `a[]` where `a[i]` represents element i.
- If 1, `a[i]` in subset; if 0, `a[i]` not in subset.

i	binary	subset	complement
0	0 0 0 0	empty	4 3 2 1
1	0 0 0 1	1	4 3 2
2	0 0 1 0	2	4 3 1
3	0 0 1 1	2 1	4 3
4	0 1 0 0	3	4 2 1
5	0 1 0 1	3 1	4 2
6	0 1 1 0	3 2	4 1
7	0 1 1 1	3 2 1	4
8	1 0 0 0	4	3 2 1
9	1 0 0 1	4 1	3 2
10	1 0 1 0	4 2	3 1
11	1 0 1 1	4 2 1	3
12	1 1 0 0	4 3	2 1
13	1 1 0 1	4 3 1	2
14	1 1 1 0	4 3 2	1
15	1 1 1 1	4 3 2 1	empty

Enumerating subsets: natural binary encoding

Given N elements, enumerate all 2^N subsets.

- Count in binary from 0 to $2^N - 1$.
- Maintain array `a[]` where `a[i]` represents element i.
- If 1, `a[i]` in subset; if 0, `a[i]` not in subset.

Binary counter from warmup does the job.

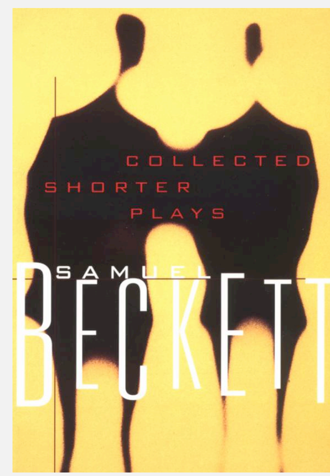
```
private void enumerate(int k)
{
  if (k == N)
  {  process(); return;  }
  enumerate(k+1);
  a[k] = 1;
  enumerate(k+1);
  a[k] = 0;
}
```

Digression: Samuel Beckett play

Quad. Starting with empty stage, 4 characters enter and exit one at a time, such that each subset of actors appears exactly once.

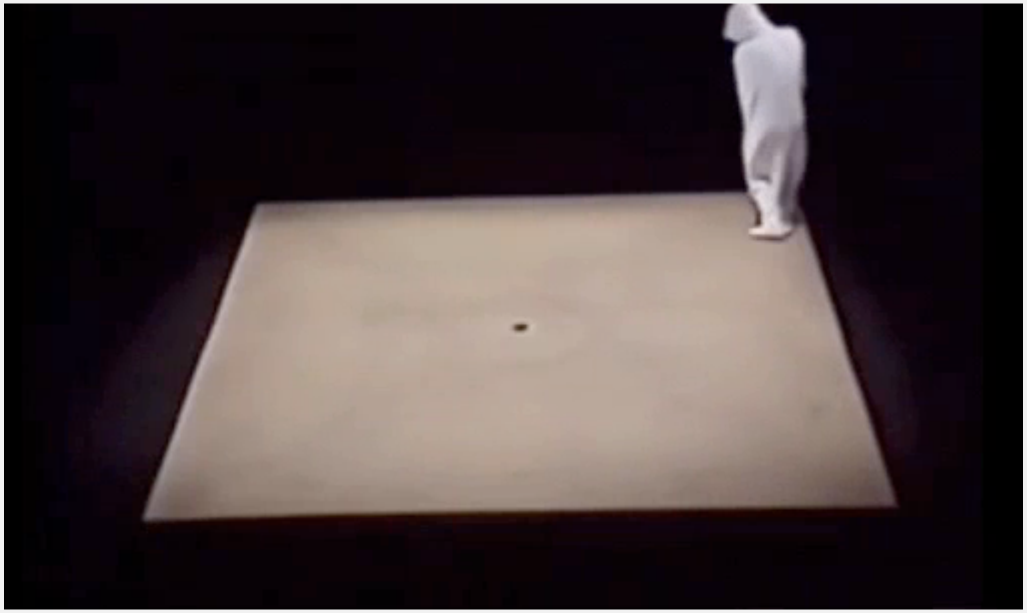
code	subset	move
0 0 0 0	*empty*	
0 0 0 1	1	enter 1
0 0 1 1	2 1	enter 2
0 0 1 0	2	exit 1
0 1 1 0	3 2	enter 3
0 1 1 1	3 2 1	enter 1
0 1 0 1	3 1	exit 2
0 1 0 0	3	exit 1
1 1 0 0	4 3	enter 4
1 1 0 1	4 3 1	enter 1
1 1 1 1	4 3 2 1	enter 2
1 1 1 0	4 3 2	exit 1
1 0 1 0	4 2	exit 3
1 0 1 1	4 2 1	enter 1
1 0 0 1	4 1	exit 2
1 0 0 0	4	exit 1

ruler function



Digression: Samuel Beckett play

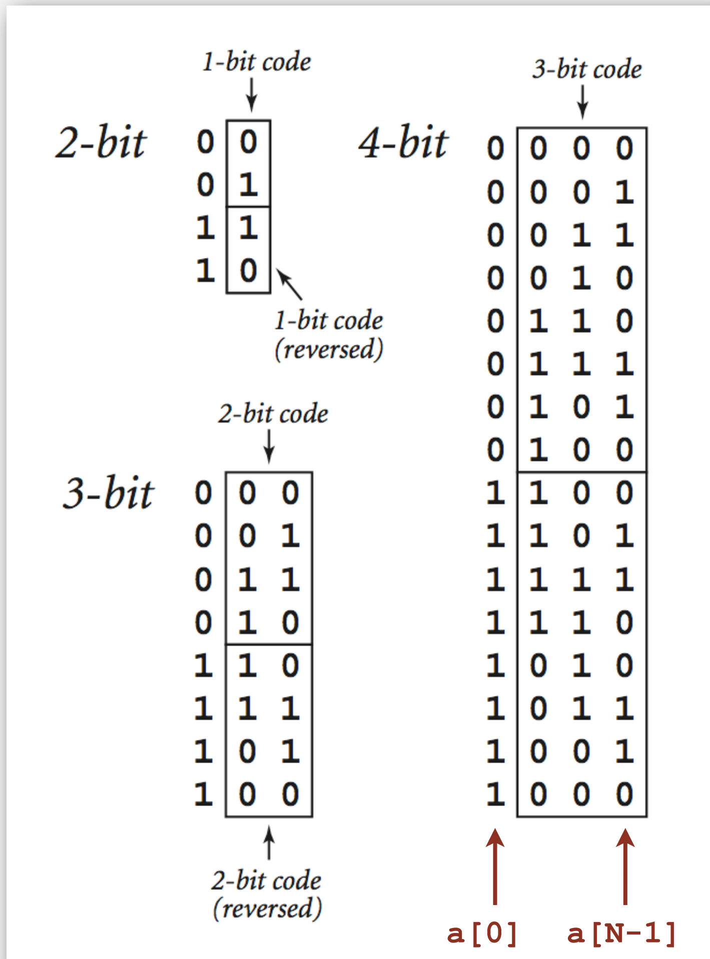
Quad. Starting with empty stage, 4 characters enter and exit one at a time, such that each subset of actors appears exactly once.



" faceless, emotionless one of the far future, a world where people are born, go through prescribed movements, fear non-being even though their lives are meaningless, and then they disappear or die." — Sidney Homan

Def. The k-bit binary reflected Gray code is:

- The $(k-1)$ bit code with a 0 prepended to each word, followed by
- The $(k-1)$ bit code in reverse order, with a 1 prepended to each word.



Enumerating subsets using Gray code

Two simple changes to binary counter from warmup:

- Flip `a[k]` instead of setting it to 1.
- Eliminate cleanup.

Gray code binary counter

```
// all bit strings in a[k] to a[N-1]
private void enumerate(int k)
{
   if (k == N)
   {  process(); return;  }
   enumerate(k+1);
   a[k] = 1 - a[k];
   enumerate(k+1);
}
```

0	0	0
0	0	1
0	1	1
0	1	0
1	1	0
1	1	1
1	0	1
1	0	0

same values
since no cleanup

standard binary counter (from warmup)

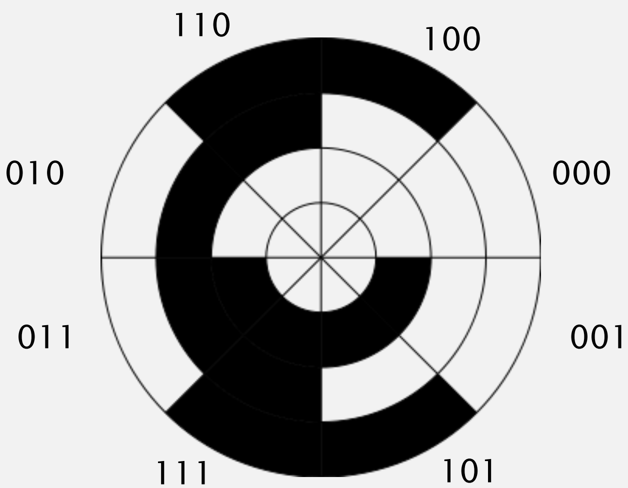
```
// all bit strings in a[k] to a[N-1]
private void enumerate(int k)
{
   if (k == N)
   {  process(); return;  }
   enumerate(k+1);
   a[k] = 1;
   enumerate(k+1);
   a[k] = 0;
}
```

0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

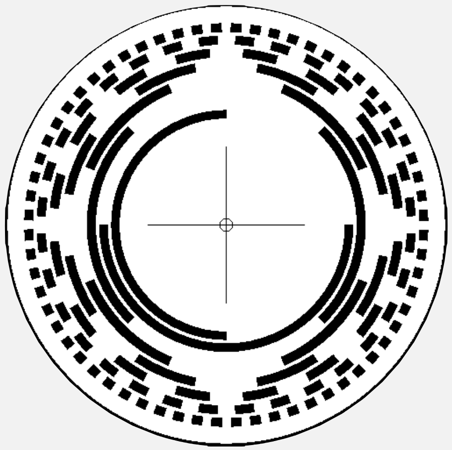
a[0]   a[N-1]

Advantage. Only one element in subset changes at a time.

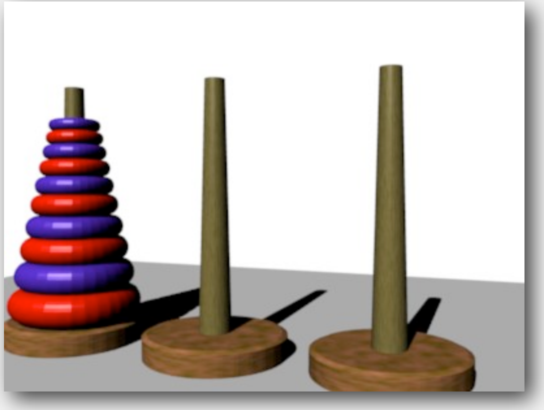
More applications of Gray codes



3-bit rotary encoder

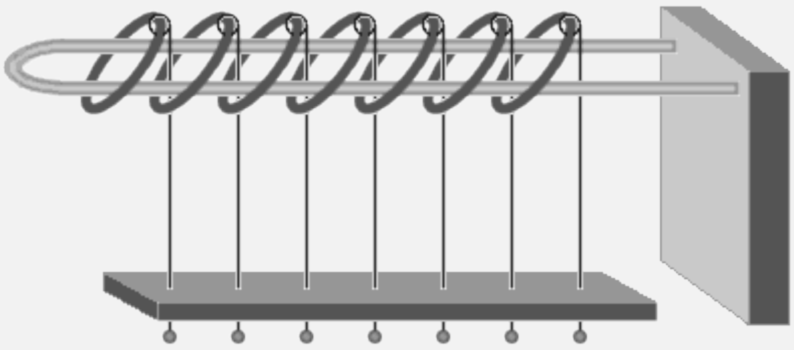


8-bit rotary encoder



Towers of Hanoi

(move ith smallest disk when bit i changes in Gray code)



Chinese ring puzzle (Baguenaudier)

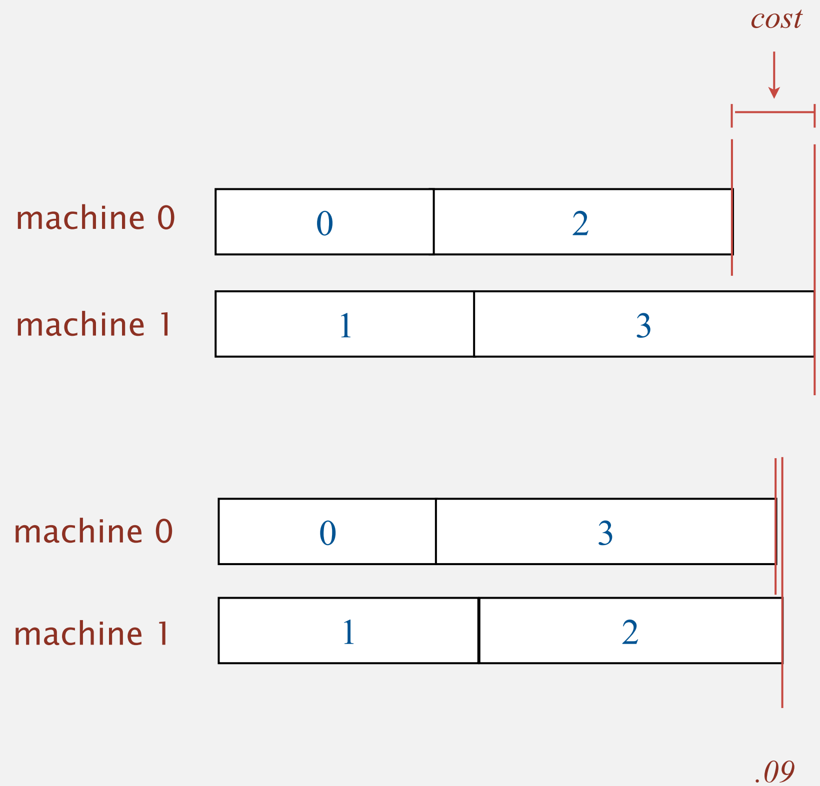
(move ith ring from right when bit i changes in Gray code)

Scheduling

Scheduling (set partitioning). Given N jobs of varying length, divide among two machines to minimize the makespan (time the last job finishes).

or, equivalently, difference between finish times

job	length
0	1.41
1	1.73
2	2.00
3	2.23



Remark. This scheduling problem is NP-complete.

Scheduling: improvements

Brute force. Enumerate 2^N subsets; compute makespan of each; return best.

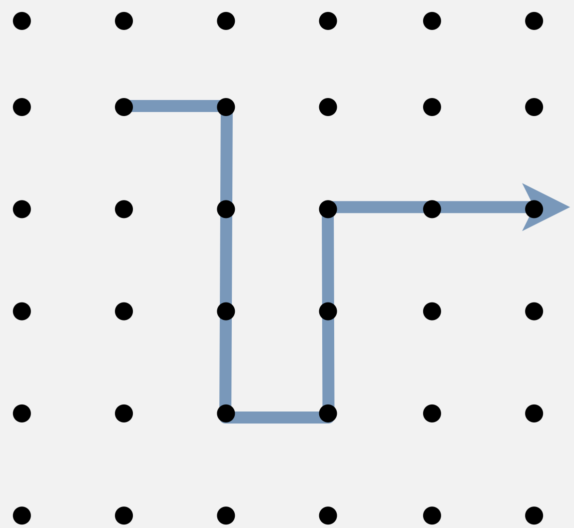
Many opportunities to improve.

- Fix first job to be on machine 0. ← factor of 2 speedup
- Maintain difference in finish times. ← factor of N speedup (using Gray code order)
  (and avoid recomputing cost from scratch)
- Backtrack when partial schedule cannot beat best known. ← huge opportunities for improvement on typical inputs
- Preprocess all 2^k subsets of last k jobs; cache results in memory. ← reduces time to 2^{N-k} at cost of 2^k memory

```
private void enumerate(int k)
{
   if (k == N) { process(); return;  }
   if (canBacktrack(k)) return;
   enumerate(k+1);
   a[k] = 1 - a[k];
   enumerate(k+1);
}
```

- permutations
- backtracking
- counting
- subsets
- **paths in a graph**

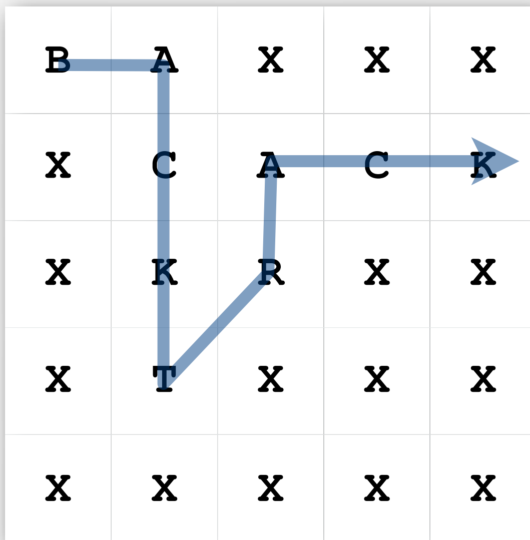
Goal. Enumerate all simple paths on a grid of adjacent sites.



no two atoms can occupy same position at same time

Application. Self-avoiding lattice walk to model polymer chains.

Boggle. Find all words that can be formed by tracing a simple path of adjacent cubes (left, right, up, down, diagonal).



Backtracking. Stop as soon as no word in dictionary contains string of letters on current path as a prefix ⇒ use a trie.

B
BA
BAX

Boggle: Java implementation

string of letters on current path to (i, j)

```
private void dfs(String prefix, int i, int j)
{
   if ((i < 0 || i >= N) ||
       (j < 0 || j >= N) ||
       (visited[i][j])   ||
       !dictionary.containsAsPrefix(prefix))
      return;

   visited[i][j] = true;
   prefix = prefix + board[i][j];

   if (dictionary.contains(prefix))
      found.add(prefix);

   for (int ii = -1; ii <= 1; ii++)
      for (int jj = -1; jj <= 1; jj++)
         dfs(prefix, i + ii, j + jj);

   visited[i][j] = false;
}
```

backtrack

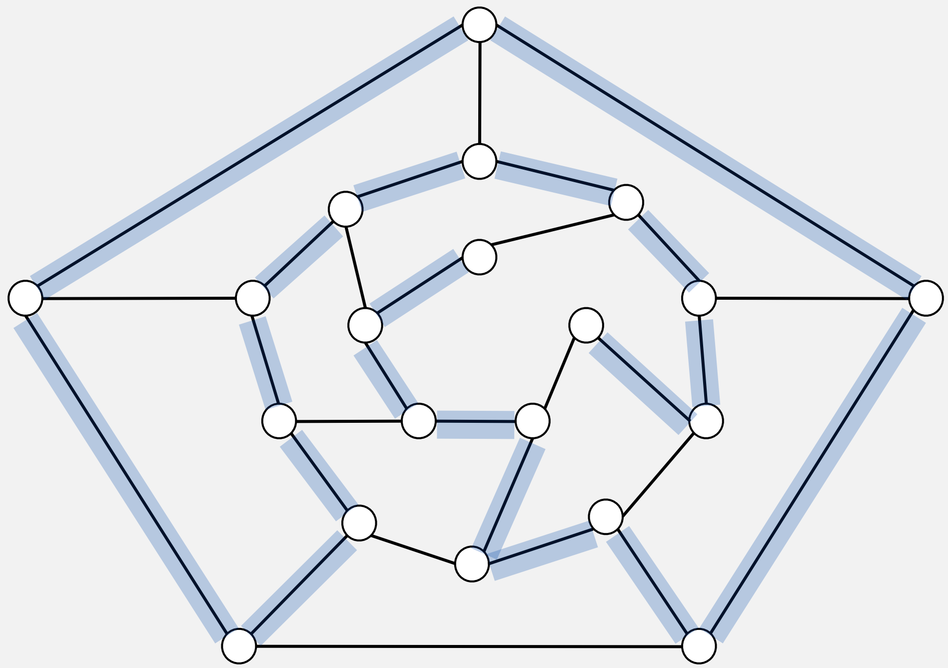
add current character

add to set of found words

try all possibilities

clean up

Goal. Find a simple path that visits every vertex exactly once.

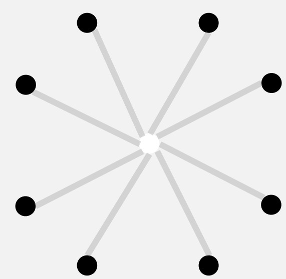


visit every edge exactly once

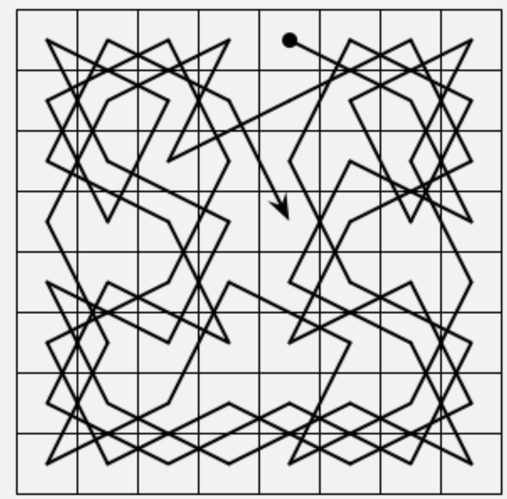
Remark. Euler path easy, but Hamilton path is NP-complete.

Knight's tour

Goal. Find a sequence of moves for a knight so that (starting from any desired square) it visits every square on a chessboard exactly once.



legal knight moves



a knight's tour

Solution. Find a Hamilton path in knight's graph.

Hamilton path: backtracking solution

Backtracking solution. To find Hamilton path starting at v :

- Add v to current path.
- For each vertex w adjacent to v
  - find a simple path starting at w using all remaining vertices
- Clean up: remove v from current path.

Q. How to implement?

A. Add cleanup to DFS (!!)

Hamilton path: Java implementation

```
public class HamiltonPath
{
   private boolean[] marked;     // vertices on current path
   private int count = 0;        // number of Hamiltonian paths

   public HamiltonPath(Graph G)
   {
      marked = new boolean[G.V()];
      for (int v = 0; v < G.V(); v++)
         dfs(G, v, 1);
   }

   private void dfs(Graph G, int v, int depth)
   {
      marked[v] = true;
      if (depth == G.V()) count++;

      for (int w : G.adj(v))
         if (!marked[w]) dfs(G, w, depth+1);

      marked[v] = false;
   }
}
```

length of current path (depth of recursion)

found one

backtrack if w is already part of path

clean up

Exhaustive search: summary

problem	enumeration	backtracking
N-rooks	permutations	no
N-queens	permutations	yes
Sudoku	base-9 numbers	yes
scheduling	subsets	yes
Boggle	paths in a grid	yes
Hamilton path	paths in a graph	yes